Initial Feasible Solution To The Transportation Problem : Composite Approximation Method

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INTRODUCTION

The Transportation Problem (TP) is a special case of Linear Programming Problem (LPP), and the algorithm is primarily designed to solve the minimization problem. TP involves the allotment of a homogenous product (or activity) from sources (factories, warehouses, etc.) to Sinks (retail shops, warehouses, destinations, etc.). Although generically known as a *'Transportation Problem'* and considered as a problem solving method for transporting goods or for designing a schedule for shipping cargo, the model can also be used for solving problems of allocation of homogeneous resources to activities, financial allocations, investment decisions, manpower allocation for various businesses, etc. Like the LPP, TP also aims at either maximization (of profits or benefits) or minimization (of costs or distance or some other quality) as a single objective, subject to the resource constraints. The TP has a special structure. The LPP solution methods like Simplex are unsuitable, complex or inefficient to solve the problems mentioned above. The special algorithms of TP enable solving such problems with ease and in much less time.

SOLVING THE TRANSPORTATION PROBLEM

The typical transportation problem involves allotting the products from *'m'* sources (origins) to *'n'* sinks (destinations) such that the total transportation cost is least. Since there are *'m'* constraint equations for availability at the sources and *'n'* constraint equations for the requirement of destinations, we have in all $(m + n)$ constraint equations to solve for the desired allotment. However, since after balancing the problem, the total availability is equal to the total requirements, one of the constraints can be written as a linear combination of other constraints using the equation that sum of all rows is equal to sum of all the columns. Hence, any solution would have only $(m + n - 1)$ basic variables. All other variables are non-basic, with their values as zero.

INITIAL FEASIBLE SOLUTION

For any simplex or any other optimization algorithm, Initial Feasible Solution (IFS) is needed as a starting point. Once IFS is obtained, optimization algorithms improve the solution step by step by maintaining the feasibility. The solution obtained is considered optimal when a state is reached where no further improvement is possible. There are two popular algorithms for optimization, MODI and Stepping Stone. Both methods use the same principle, in which the effect of changing allotments of IFS to each unoccupied cell is tested. If change in allotment reduces the overall transportation cost, the allotment is shuffled to affect the cost reduction. The above process of checking for feasibility of cost reduction is again repeated. Once all the unoccupied cells are tested, and no further improvement is feasible, the optimum solution is obtained. Since all the unoccupied cells should be tested (i.e. Non-Basic Variables) for possible improvement to the existing solution, the optimization techniques require a lot of computational efforts. There are $(m \times n)$ variables, of which $m + n - 1$ are basic. Thus, the number of non-basic variables are $(m - 1) \times (n - 1)$. As the number of sources and destinations increase, the number of cells to be tested for improvement increases very rapidly. For example, if there are three sources and three destinations, four cells per iteration should be tested. This number increases to sixteen if there are five sources and five destinations. Hence, efficient IFS should be as near to the optimum solution as possible to save the number of iterations required for arriving at the optimal solution, thereby reducing the computational efforts. There are many methods to find IFS. In all these methods, one of the cells is selected as per the allotment criteria, and the maximum possible quantity is allotted, subject to availability and

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requirement limitations. After each allotment, the exhausted row or column is not considered anymore. Then, the next possible cell as per the criteria is selected for the allotment. Finally, for a balanced problem, when a single row and a single column remain for the allotment, the row and the column (availability and demand) would get exhausted together. Thus, there would be $(m + n - 1)$ allotments. The methods used for IFS differ only in the cell selection criteria for the allotments.

Most commonly used methods for finding IFS are North West Corner Rule, Least Cost Method or Matrix Minima Method, Vogel Approximation Method (VAM) and Russell Approximation Method (RAM). North West Corner Rule is the simplest of all these methods for finding an initial feasible solution. In this method, allocations are made without considering the transportation cost. The top-left (North-west) corner cell is selected, and the maximum possible quantity is allotted to that cell, irrespective of the cost in that cell. Matrix Minima Method or Least Cost Method considers the cost of transportation in each cell. The principle behind this method is to select the cell having the least cost and allot as much quantity as possible within the constraints of availability at the source and requirement of the destination.

A similar principle is used in the Column Minima Method and Row Minima Method. Vogel Approximation Method (VAM) is by far the most efficient method to find the initial feasible solution and hence, is popularly used. Although it involves more computational exercise as compared to other methods, the initial feasible solution obtained by VAM is likely to be closer to the optimum solution. This saves considerable computational iterations during optimization routines. VAM is a heuristic method that measures loss of opportunity in terms of the penalty one has to pay if he does not use the lowest-cost cell in every column and row. The lowest-cost cell from a row or column with maximum penalty is selected. Russell's Approximation Method (RAM) is computationally complex, but usually gives the best IFS.

COMPOSITE APPROXIMATION METHOD FOR FINDING THE INITIAL FEASIBLE SOLUTION TO THE TRANSPORTATION PROBLEM

The researchers propose a simpler method that gives a better IFS. It uses two principles (concepts) viz. *'Matrix Reduction'* and Penalty as VAM. Hence, the researchers have named it as *'Composite Approximation Method (CAM)'*. The matrix reduction principle is, *'If a constant is added or subtracted from every element of a row or a column, the optimum solution remains the same'*. If the transportation cost to all destinations is reduced by the same amount from a given source, the decision would not change. The optimum route would remain the same. Similarly, if cost to any destination from every source is reduced by the same amount, the best route would remain the same. Hence, the cost matrix is reduced by subtracting the minimum row cost from all cost coefficients (including itself) of that row; and the minimum column cost from all the cost coefficients of that column (including itself). This reduced-cost matrix is simple to handle as there will be many zeros as cost coefficients. If we use the cells with zero costs for allotment, obviously, we would get a solution closer to the optimum. Since there are many zero cost cells competing for allotments, we would calculate the penalty associated with every cell with zero cost. The penalty indicates additional unit cost in the row or column that we will have to pay if we do not use the cell for allotment. This method is an extension of the idea suggested by Prof . R. Sivakumar and Prof. P. R. Vittal. It has been observed that the initial feasible solution obtained by this method is better than all other methods and in many cases, it gives an optimum solution. On the other hand, the method is much simpler and requires a lesser computational exercise than VAM or RAM. The procedure is as follows:

❅**Step 1 : Balance The Problem :** First, write the cost (or profit) coefficient table as a matrix. If the total supply is equal to the total demand, it is a balanced problem. If the problem is balanced, proceed to the next step. If the problem is not balanced, add a dummy row or column with Supply or Demand equal to the difference in availability and requirement. The transportation cost is zero for cells of a Dummy. This is obvious because the dummy origin or dummy destination does not exist and hence, there is no transportation per se, and it costs nothing.

❅**Step 2: Invert The Maximization Problem To Minimization, If Applicable :** The TP solution algorithm, as mentioned earlier, is primarily meant to solve the minimization problems, however, it can be used to solve the maximization problems also (with slight modification). If the problem is of the maximization type, identify the highest profit coefficient in the profit matrix, and subtract all the profit coefficients of the respective cells from that highest value. Re-write the new matrix and proceed to Step 3. If the problem is already of the minimization type, proceed to the Step 3. (Note: It is possible to solve the maximization problem directly. But such method becomes unnecessarily complicated. So, the researchers use the method given in the Step 2 above.)

❁**Step 3: Row and Column Reduction :** For the row reduction, find out the smallest cost in every row and subtract it from each element of that row (including itself). Repeat it for all rows. Thus, in each row, there would now be at least one zero element. The new table thus obtained is called row reduced matrix or first reduced table. Then repeat the same procedure for the columns using the first reduced table. Find the smallest cost in every column of the row reduced matrix. Subtract this from each element of the corresponding column (including itself). As a result, every column would now have at least one zero element. This new table is called as reduced matrix or second reduced table. We can either perform row reduction followed by a column reduction, or we can first perform column reduction followed by a row reduction. After the row and column reduction, every row and column has, at least, one zero. Hence, further reduction is not possible.

❁**Step 4 : Calculation Of Penalties For The Cells With Zero Elements After Reduction :** The researchers selected the cells with zero reduced costs and found penalties for these cells. For calculating the penalty for a cell, check all the reduced costs in the row and column in which the cell is at the intersection. Row penalty is equal to the next higher element in that row. If the row has another zero, the penalty is zero. Similarly, the column penalty is equal to the next higher element in that column. If the column has another zero, the penalty is zero. For the selected cell, the higher among the row and column penalties should be noted. Note that the penalty calculation is the same as VAM, but in this case, the researchers calculated the penalties only for the cells with zero reduced cost, and they took the penalty for the cell as the higher value between the row and the column penalties.

❁ **Step 5 : Transportation Allotments :** Now, first allot the maximum possible quantities in the cells with reduced cost zero, which has the highest penalty as compared to other zero cost cells. Remove the row or column which has exhausted its demand or supply (same as other methods). Adjust the demand / supply accordingly. Repeat the process for the remaining zero cells. In case two zero cells have the same highest penalty (i.e. if there is a tie), select the cell where a larger quantity can be allotted. If again, there is a tie for a quantity to be allotted, check the next level penalty, which is the difference between the lowest element and third lowest element in the row and column of that cell. Once all the cells with zero reduced costs are allotted or removed, select the cell with the least reduced cost in the residual matrix for allotments. In case of a tie of cost, follow the process of comparing penalties. In case of a tie between the penalties, select the cell between them where the maximum quantity could be allotted. If this is also equal, check the next level penalty, which is the difference between the lowest element and the third lowest element in the row and column of that cell. Repeat the process on the residual matrix until all the allotments are done. This way, the Initial Feasible Solution obtained is efficient.

❁ **Illustrative Example 1:** A company is transporting its products from three plants to four distribution centers. The supply, demand and unit costs of transportation are shown in the Table 1. Find the initial feasible solution.

Table 1: Unit Transportation Cost in ₹

		Distribution Center				
		1	2	3	4	Supply
	1	20	30	50	17	7
Plant	2	70	35	40	60	10
	3	40	12	60	25	18
	Demand	5	8	7	15	35

❁**Solution:** The problem is balanced. It can be seen that the IFS by different methods is as follows :

By Least Cost Method (LCM): ₹985

By VAM: ₹940

The VAM initial feasible solution is better (closer to optimum) than LCM.

The Initial feasible solution by the proposed Composite Approximation Method is found as follows: First for row

Table 2: Row Reduced Matrix						
		Distribution Center				Supply
		1	2	3	4	
Plant	1	3	13	33	0	7
	2	35	0	5	25	10
	3	28	0	48	13	18
	Demand	5	8	7	15	35

reduction, subtract the lowest cost in every row from all the elements of that row. For example, the lowest element in the first row is 17. So we subtract it from all elements of the first row. Similarly, for the second row, subtract 35, which is the lowest element in that row, and so on. We get the row reduced matrix or the first reduced table as shown in Table 2. Now, use this table for column reduction using a similar procedure. For column reduction, subtract the lowest element in every column of the row reduced matrix from all the elements of that column. For example, the lowest element in the first column is 3. So, subtract it from all elements of the first column. Similarly, for the second column, the lowest element is 0 , that we subtract from all elements of the second column, and so on. We get the column reduced matrix as or second reduced table as follows (Table 3) :

Table 3: Column Reduced Matrix						
		Distribution Center				Supply
		1	2	3	4	
Plant	1	0	13	28	0	7
	2	32	0	0	25	10
	3	25	0	43	13	18
	Demand	5	8	7	15	35

Penalties are calculated for the cells where the reduced cost of transportation is zero.

Table 4: Penalties						
		Distribution Center				Supply
		1	2	3	4	
Plant	1	0 △25	13	28	0 △13	**7**
	2	32	0 △0	0 △28	25	**10**
	3	25	0 △13	43	13	**18**
	Demand	5	8	7	15	

The researchers calculated the penalty for the cell as row and column wise and took the higher value between them. The penalties are shown in the Table 3. The highest penalty is 28 for cell (2, 3). Hence, allot as much quantity as possible i.e. 7 in cell (2, 3) which exhausts the demand of the distribution centre 3 (column 3). After adjusting the demand and supply quantities after allotment, penalties are recalculated for zero cost cells and are shown in the Table 4. Note that the column 3 is removed and hence, is not used for penalty calculations.

Table 5: First Allotment						
		Distribution Center				Supply
		1	2	3	4	
Plant	1	0 △25	13	28	0 △13	7
	2	32	0 △28	0 (7)	25	10
	3	25	0 △13	43	13	18
	Demand	5	8	-	15	

Penalties are calculated for zero cost cells. Cell penalties are higher value between row and column penalties for that cell. For the highest penalty, there is a tie. So as per the procedure, the researchers look for the cell where the highest allotment is possible among them. The cell (1, 1) can be allotted quantity 5, and the cell (2, 2) can be allotted the quantity 3. So select cell (1, 1) with the maximum possible allotment, which in this case is 5. The allotment fulfills the demand of center 1. After allotment, readjustment of demand and supply is carried out, and penalties are calculated as per the procedure. The results are shown in the Table 5.

Table 6: Second Allotment

		Distribution Center				
		1	2	3	4	Supply
Plant	1	0 (5)	13	28	0 △13	2
	2	32	0 △28	0 (7)	25	8
	3	25	0 △13	43	13	18
	Demand	-	8	-	15	

Now, the highest penalty cell is (2, 2), with a penalty of 25 (Note that its row penalty is 25 and the column penalty is zero. Hence, cell penalty is 25). Allot maximum possible quantity of 3. This would exhaust the demand of plant 2. After allotment, readjustment and recalculated penalties, the matrix becomes as follows (see Table 7).

Table 7: Third Allotment

		Distribution Center				
		1	2	3	4	Supply
Plant	1	0 (5)	13	28	0 △13	2
	2	32	0 (3)	0 (7)	25	-
	3	25	0 △13	43	13	18
	Demand	-	5	-	15	

Table 8: Allotments' Initial Feasible Solution

		Distribution Center				
		1	2	3	4	Supply
Plant	1	5			2	7
	2		3	7		10
	3		5		13	18
	Demand	5	8	7	15	

Table 9: Initial Feasible Solution By CAM

From Plant	To Distribution Center	Quantity	Cost per Unit	Total Cost
1	1	5	20	100
1	4	2	17	34
2	2	3	35	105
2	3	7	40	280
3	2	5	12	60
3	4	13	25	325
Total Cost of Transportation Initial Feasible Solution				**904**

There is a tie for the highest penalty of 13. As per the procedure, we select cell (3, 2) as it can be allotted more quantity than cell (1, 4). So allot maximum possible quantity of 5 in cell (3, 2). This meets the demand of Centre 2. After this allotment, we have only one column left for allotment and hence, there is only one feasible solution way of allotment as quantity 2 in cell (1, 4) and quantity 13 in cell (3, 4). Thus, the initial feasible solution is shown in the Table 7. *The solution is much better than VAM. In fact, the Initial Feasible Solution is indeed the Optimum Solution!!*

☸ **Illustrative Example 2:** Solve the following transportation problem to get the Initial Feasible Solution by all the methods, compare the solutions and check optimality.

Table 10: Unit Transportation Cost

	A	B	C	D	Supply
X	2	3	11	7	6
Y	1	0	6	1	1
Z	5	8	15	9	10
Requirement	7	5	3	2	17

☸ **Solution By Composite Approximation Method:** After row and column reduction, the matrix reduces to the following reduced cost matrix or second reduced table (see Table 11).

Table 11: Reduced Cost Matrix Or Second Reduced Table

	A	B	C	D	Supply
X	0	1	3	4	6
Y	1	0	0	0	1
Z	0	3	4	3	10
Requirement	7	5	3	2	17

Now, complete the allotments as per the explained Composite Approximation Method. First, calculate the penalties for the cells where the reduced cost of transportation is zero. Calculate the penalty for the cell as row and column wise and take the higher value between them. The penalties are shown in the Table 12.

Table 12: Penalties for Reduced Zero Cost

	A	B	C	D	Supply
X	0 △1	1	3	4	6
Y	1	0 △1	0 △3	0 △3	1
Z	0 △3	3	4	3	10
Requirement	7	5	3	2	17

There is a tie for the highest penalty 3 for cells (Y, C), (Y, D) and (Z, A). Select the cell (Z, A) i.e. (3, 1), since we can

Table 13: Penalties For Reduced Zero Cost Cell After First Allotment

	A	B	C	D	Supply
X	0 –	1	3	4	6
Y	1 –	0 △1	0 △3	0 △3	1
Z	0 (7)	3	4	3	10
Requirement	7	5	3	2	17

allot maximum quantity 7, which is larger than the possible quantities in other competing cells. Allotment of quantity 7 in cell (Z, A) exhausts the requirement of column A. After adjusting the requirement and supply quantities on this allotment, penalties are recalculated for zero cost cells and are shown in Table 13. Note that the column A (i.e. first column) is removed and hence, is not used for penalty calculations.

Table 14: Final Allotment Using Composite Approximation Method					
	A	B	C	D	Supply
X		(5)	(1)		6
Y			(1)		1
Z	(7)		(1)	(2)	10
Requirement	7	5	3	2	17

There is a tie for the highest penalty 3 for cells (Y, C) and (Y, D). In both the cells, we can allot maximum quantity of 1. Hence, as per the suggested procedure, we select cell (Y, C) arbitrarily. Allotting as much quantity as possible i.e. 1 in cell (Y, C) exhausts the supply of row 2. After adjusting the demand and supply quantities after allotment and removing the row 2 (i.e. Y), there is no cell with zero reduced cost. Now, the lowest cost cell is (A, B) with cost 1. Allot maximum possible quantity 5. This removes the column B (i.e. second column). Next, lowest penalties are 3 for cells (A, C) and (Z, D). In any case, we need to allot maximum possible quantities in both cells, which ever sequence we follow. Finally, we allot only possible cell (Z, C), the quantity 1. Thus, the final allotment is given in the Table 14.

Table 15: Summary Of Initial Feasible Solutions By Different Methods And Comments		
Method	**Cost of Transportation**	**Remarks**
North West Corner Rule	116	Solution is not good. It is degenerate.
Least Cost Method	112	The solution is far from the Optimum.
Vogel Approximation Method	102	The solution is closest to the Optimum.
Russell's Approximation Method	103	The solution is good. It is degenerate.
Composite Approximation Method	100	The solution is the best and is indeed Optimum.

Total transportation cost is ₹ 100. The optimum (minimum) total transportation cost is ₹100.
6.009
3.504

COMPARISON OF VARIOUS METHODS FOR INITIAL FEASIBLE SOLUTION

a) The North West Corner rule is the simplest of all, but usually provides inferior Initial Feasible Solution. Hence, it needs more number of iterations for optimum solution.

b) Column Minima, Row Minima and Matrix Minima Method (Least Cost Method) are computationally simpler and better as compared to the North West Corner Method.

c) VAM is presently the most popular method for manual calculation. It is efficient and relatively easy to implement. It is conceptually superior to the Matrix Minima Method.

Table 16: Evaluation of CAM	
Total Problems solved.	21
Initial Feasible Solution by CAM is Optimum solution.	15 (71.4%)
Solution found better than VAM.	8 (38.1%)
Solution is Same as VAM.	11 (52.4%)
Solution is inferior to VAM.	2 (9.5%)

d) RAM is an excellent method that is efficient and quick to implement on a computer. However, it is tedious and time consuming for manual calculations. Although we cannot say with certainty which is the better method as far as effectiveness of solution is concerned, usually, the RAM provides a better solution than VAM.

e) Composite Approximation Method is based on matrix reduction, and is computationally simple and is the most efficient of all the methods. In most cases, it gives the Initial Feasible Solution, which is optimum. In almost all cases, the solution is better than the other methods.

TRIAL EVALUATION OF THE METHOD

CAM was used to find the initial feasible solution for twenty one randomly selected transportation problems. Out of the twenty one problems, in fifteen problems, the initial feasible solution by CAM was actually the optimum. In eight problems, the initial feasible solution by CAM was better than VAM; in eleven cases, it was the same as VAM, and only in two cases, it was slightly inferior to the VAM. Thus, it is observed that the proposed Composite Approximation is computationally superior and is simple as compared to the presently used methods for finding Initial Feasible Solution of the Transportation Problem.

Thus, Composite Approximation Method reduces the number of iterations for finding the optimum solution to the transportation problem. It may be further investigated whether this proposed algorithm could be used for computer-based optimization.

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